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Perturbation of infra-red fixed points and duality in quantum impurity problems.

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We explain in this paper how a meaningful irrelevant perturbation theory around the infra-red (strong coupling) fixed point can be carried out for integrable quantum impurity problems. This is illustrated in details for the spin 1/2 Kondo model, where our approach gives rise to the complete low temperature expansion of the resistivity, beyond the well known T^2 Fermi liquid behaviour. We also consider the edge states tunneling problem, and demonstrate by Keldysh techniques that the DC current satisfies an *exact duality* between the UV and IR regimes. This corresponds physically to a duality between the tunneling of Laughlin quasi particles and electrons, and, more formally, to the existence of an exact instantons expansion. The duality is deeply connected with integrability, and could not have been expected a priori.

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1. Introduction

Duality arguments have been commonly used in quantum impurity problems for many years. An archetypal situation is provided by the model of a particle moving in a periodic potential and subject to quantum dissipation [1]. This problem is represented in one dimension [1] by the following (boundary sine-Gordon or BSG) hamiltonian

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[(\partial_x \Phi)^2 + (\Pi)^2\right] + 2\lambda \cos\sqrt{2\pi g}\Phi(0). \tag{1.1}$$

where the field at the origin represents the particle coordinate, and the bulk free boson the bath degrees of freedom. At small λ the particle diffuses freely (UV fixed point) , while at large λ, it is localized (IR fixed point) in a minimum of the potential, given by $\Phi(0) = \sqrt{\frac{2\pi}{g}}n$, n an integer. Near the UV fixed point, physical properties can be expanded in powers of λ, and are expressed in terms of Coulomb gas integrals whose charges $\pm g$ correspond to the two possible exponentials in the cosine term (the dimension of the operator $\cos\sqrt{2\pi g}\Phi(0)$ being $\Delta = g$). It is also possible to study the vicinity of this IR fixed point in an $\frac{1}{\lambda}$ expansion by considering the instantons and anti-instantons that take the particle from one minima to a neighbouring one. Using the leading order action of these instantons, one obtains again a Coulomb gas, but this time with charges $\pm\frac{1}{g}$. This demonstrates, in slightly more formal terms, that the leading IR hamiltonian looks as (1.1), but with a perturbation $\lambda_d \cos\sqrt{\frac{2\pi}{g}}\tilde{\Phi}$ (with dimension $\Delta_d = \frac{1}{g}$), where $\tilde{\Phi}$ is the dual of the free boson in the usual sense, and by dimensional analysis, $\lambda_d \propto \lambda^{-\frac{1}{g}}$.

The same hamiltonian (1.1) appears also in the problem of tunneling between edge states in the fractional quantum Hall effect [2]. In that case, while it is Laughlin quasi particles of charge $g = \nu$ (the filling fraction) that tunnel in the UV, the duality argument demonstrates that it is electrons of charge unity that tunnel in the IR.

The duality just discussed is very useful qualitatively. It has however been used in the literature as a much stronger statement: namely that physical properties should exhibit an *exact* duality between the UV and IR fixed points under replacement of λ by λ_d and g by $\frac{1}{g}$. Why this should be the case was not explained, and it must be stressed that this is a highly non trivial result: the approach to the IR (strong coupling) fixed point is, in general, determined by a very specific combination of irrelevant operators coming with amplitudes that are all powers of λ_d, so they all contribute equally significantly: for

[1] Space dimensionality does not play a crucial role here

instance, one expects that, in addition to the term $\lambda_d \cos\sqrt{\frac{2\pi}{g}}\tilde{\Phi}$, terms $\lambda_d^{\frac{n^2\Delta_d-1}{\Delta_d-1}}\cos n\sqrt{\frac{2\pi}{g}}\tilde{\Phi}$ should also appear (where $\Delta_d=\frac{1}{g}$), corresponding physically to multi-instantons processes, or tunneling of several electrons. Such terms might also be required as counterterms to cure the very strong short distance divergences of the IR perturbation theory. Clearly, the existence of these terms will destroy any hope of observing an exact duality, and one should not expect the duality argument to tell us more than the leading irrelevant operator, in a general situation.

Nevertheless, an analytical computation of the mobility (the current) at $T=0$ and with an external force (bias) [3] has exhibited an exact duality between the UV and IR for the model (1.1), adding up confusion to the whole issue. Something very special must be happening in that case - and indeed, the model is integrable.

One of the purposes of this paper is to discuss why integrability gives rise to an exact duality for some physical properties - and also, to explain why this duality should not be expected for other properties. In discussing these questions, we will actually consider IR perturbation theory, and show how it can be made meaningful, again thanks to integrability. This has applications beyond the tunneling problem: as an example, we discuss in details the case of the resistivity in the Kondo problem.

In the second section of this paper, we use the simple example of the Ising model with a boundary magnetic field to discuss how the integrable structure of quantum impurity problems gives a quick access to the full hamiltonian near the IR fixed point, which is essentially encoded in the reflection matrix, or the boundary free energy. We discuss the issue of regularization for the IR perturbation theory that arises in integrable models, and how one can use the renormalization group backwards in some cases.

In the third section of this paper, we discuss how the IR action can be determined for the spin 1/2 Kondo problem and for the tunneling problem. We also sketch the result for the higher spin Kondo problem.

In the fourth section of this paper, we discuss, as an application of IR perturbation theory, the resistivity in the Kondo problem - and determine, beyond the well known T^2 order, its complete low temperature behaviour.

In the fifth section of this paper, we finally discuss duality issues. We show that the anisotropic higher spin Kondo model exhibits a partial duality, manifest for instance in the following relation

$$f\left(j,\lambda,H,g\right)\equiv f\left(j-\frac{1}{2},\lambda_d,\frac{H}{g},\frac{1}{g}\right), \tag{1.2}$$

that holds up to analytical terms (odd powers) in H/T_B. We also show that the current in the tunneling problem obeys an exact duality

$$I(\lambda, g, V, T) = gV - gI\left(\lambda_d, \frac{1}{g}, gV, T\right). \tag{1.3}$$

These duality properties follow from the structure of the IR hamiltonians that is strongly constrained by integrability: within our "analytic" regularization scheme, they are made up of an infinite series of local (conserved) quantities (polynomials in derivatives of Φ), plus at most *one* non local term, which is $\lambda_d \cos\sqrt{\frac{2\pi}{g}}\tilde{\Phi}(0)$ for the BSG case, and $\lambda_d S_- e^{i\sqrt{\frac{2\pi}{g}}\tilde{\Phi}(0)} + cc$ for the spin j Kondo case (where here $S_\pm$ are spin $j-1/2$ operators) (all this within a well defined regularization scheme). As a result, thermodynamic quantities will in general exhibit a partial duality; the UV expansion in even powers of λ will match the part of the IR expansion that is in even powers of λ_d, although there will also be other terms in this IR expansion due to the local conserved quantities. Some other properties turn out to be blind to the local conserved quantities however, and as a result exhibit an *exact* duality, like the DC current in the tunneling problem.

In the first appendix, we determine the normalization of conserved quantities in the sine-Gordon theory. The second appendix contains some remarks about the Keldysh formalism and analytic continuation.

Some of the results presented here have appeared in short form in [4]. The methods we develop are related, although independent and different, to the series of works by Bazhanov, Lukyanov and Zamolodchikov [5], and also to the work of Lukyanov [6]. Duality in quantum impurity problems has also been investigated by Fendley [7], and by Fendley and one of us [8], from a more formal perspective.

2. Getting the IR hamiltonian in integrable boundary field theories: the case of the Ising model.

2.1. Some generalities

We consider the Ising model defined on the half space $x \in [-\infty, 0]$, $y \in [-\infty, \infty]$. We initially use a crossed channel or *open string* description, where euclidian time runs in the

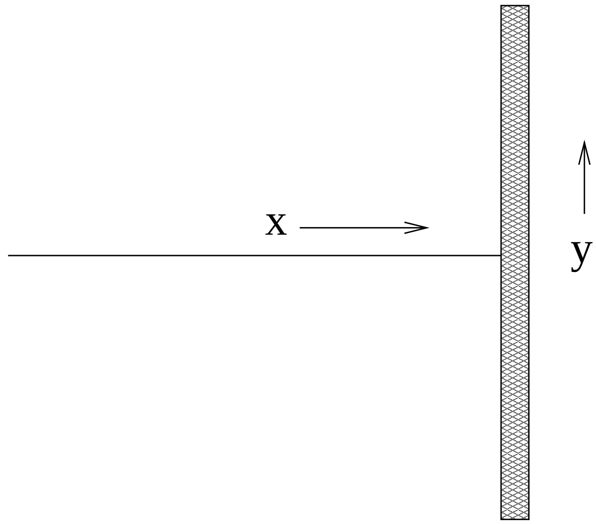

Fig. 1: Geometry of the problem.

y direction, and we introduce the complex coordinate $w = -y + ix$. We restrict to the case of a theory that is massless in the bulk, and add up a boundary magnetic field h (Fig. 1).

When $h = 0$, the fermions have free boundary conditions $\psi_L = \psi_R$ on the boundary; this is the UV fixed point. When $h \to \infty$, the Ising spins become fixed, corresponding to $\psi_L = -\psi_R$; this is the IR fixed point. The question we wish to study is how the IR fixed point is approached. More precisely maybe, we want to be able to describe the Ising model at large values of h with a hamiltonian

$$\mathcal{H} = \mathcal{H}_{IR} + \delta\mathcal{H}(0)$$

where $\mathcal{H}$ expands in some powers of the inverse coupling constant $1/h$. It is possible to gain some quick insight on what $\delta\mathcal{H}$ should look like. - it is actually an expansion in odd powers of $1/h^2$. This is because the operator content of the Ising model with fixed boundary conditions can easily be extracted from conformal invariance considerations: with fixed boundary conditions on a cylinder of length L and circumference $1/T$, the Ising partition function is simply the identity character (setting $q = e^{-\pi/LT}$)

$$\chi_0 = \frac{1}{2}\left[\prod_0^\infty \left(1 - q^{n+1/2}\right) + \prod_0^\infty \left(1 + q^{n+1/2}\right)\right].$$

From this, it follows that the only available operators are of the form $\partial^p\psi_R\partial^q\psi_R + (R \to L)$ (here, ∂ stands for ∂_w). Up to total derivatives which do not affect the physical properties

of interest, we can restrict to $\psi_R \partial^n \psi_R + (R \to L)$, with n odd. Introducing the operator (the normalization is chosen for later convenience)

$$\mathcal{O}^o_{2k+2} = (-1)^{k+1} \frac{1}{4} \left(: \psi_R \partial_w^{2k+1} \psi_R : + : \psi_L \partial_{\bar{w}}^{2k+1} \psi_L : \right), \tag{2.1}$$

we thus expect

$$\delta \mathcal{H} = \sum_{k=0}^{\infty} a_{2k+1} \frac{1}{h^{2(2k+1)}} \mathcal{O}^o_{2k+2}(0), \tag{2.2}$$

where the coefficients a_{2k+1} have to be determined. Notice that in practice, the manipulation of expressions like (2.2) will give rise to extremely strong short distance divergences - the numerical values of a_{2k+1} will only have a well defined meaning within a specific regularization scheme.

For a general problem, such a computation would appear untractable. What makes it feasible in the cases we are going to consider in this paper is integrability. To see how this comes about, and pave the way for generalizations, let us describe the Ising model using massless scattering. In this simple case, we have massless R and L moving fermionic particles with energy and momentum parametrized as $e = \pm p = e^\beta$, β the rapidity. The mode expansion of the fermion operators is

$$\begin{aligned} \psi_R(w) &= \int \frac{d\beta}{2\pi} e^{\beta/2} \left[\omega \exp(e^\beta w) Z_R(\beta) + \bar{\omega} \exp(-e^\beta w) Z_R^*(\beta) \right] \\ \psi_L(\bar{w}) &= \int \frac{d\beta}{2\pi} e^{\beta/2} \left[\bar{\omega} \exp(e^\beta \bar{w}) Z_L(\beta) + \omega \exp(-e^\beta \bar{w}) Z_L^*(\beta) \right], \end{aligned} \tag{2.3}$$

where the Z are creation and annihilation operators obeying the usual anticommuting relations, $\omega = e^{i\pi/4}$. The theory is defined on the half space $x \in [-\infty, 0]$ only; as a result, the L and R modes are not independent. Because the boundary interaction is integrable, the fermions, in the crossed channel picture scatter off the boundary one by one, with no particle production, and one has $Z_R^*(\beta) = R(\beta) Z_L^*(\beta)$, R the reflection matrix [9].

We will also use the direct channel or *closed string* picture, where euclidian time runs in the x direction. The mode expansion of the fermions is identical to (2.3), with w replaced by the variable $z = w/i = x + iy$. The Hilbert space is then the usual one for fermions defined on the whole line, and there is no relation between L and R modes. Rather, in the direct channel picture, the effect of the boundary is taken into account by the existence of a *boundary state*, which reads [9] [2]

$$|B\rangle \propto \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_i \frac{d\beta_i}{2\pi} K(\beta_i - \beta_B) Z_L^*(\beta_i) Z_R^*(\beta_i) |0\rangle, \tag{2.4}$$

[2] We do not discuss the problem of the overall normalization of the boundary state in this multiparticle description - it is enough to recall that it is independent of β_B.

with $K(\beta) = R\left(\frac{i\pi}{2} - \beta\right)$. In the simple case of a boundary magnetic field considered so far, $K(\beta - \beta_B) = i \tanh \frac{\beta_B - \beta}{2}$. The parameter β_B is in general related with a typical energy scale associated with the boundary interaction, $T_B = e^{\beta_B}$. In the case of a boundary magnetic field, $T_B \propto h^2$. In the closed string channel, we introduce the equivalent of (2.1)

$$\mathcal{O}_{2k+2} = \frac{1}{4}\left(:\psi_R \partial_z^{2k+1} \psi_R : + : \psi_L \partial_{\bar{z}}^{2k+1} \psi_L :\right). \tag{2.5}$$

2.2. The complete IR action

Let us now discuss how the IR action can be simply extracted from the knowledge of the reflection matrix, or, equivalently, of the boundary state.

To do so, let us keep working in the closed string channel, and consider the expression for the boundary state $|B>$ further. The IR boundary state (fixed boundary conditions) is obtained as $\beta_B \to \infty$ where $K = i$:

$$|B_{IR}\rangle \propto \sum_{n=0}^{\infty} \frac{i^n}{n!} \int \prod_i \frac{d\beta_i}{2\pi} Z_L^*(\beta_i) Z_R^*(\beta_i)|0> .$$

One can thus write $|B\rangle = \mathcal{B}|B_{IR}\rangle$, where the operator $\mathcal{B}$ is defined in the multiparticle basis by

$$\mathcal{B} \prod_i Z_L^*(\beta_i) Z_R^*(\beta_i)|0>= \prod_i \frac{K(\beta_i - \beta_B)}{i} Z_L^*(\beta_i) Z_R^*(\beta_i)|0> .$$

Let us expand

$$\ln\left[\frac{K(\beta - \beta_B)}{i}\right] = \sum_{k=0}^{\infty} \frac{-2}{2k+1}\ e^{(2k+1)(\beta - \beta_B)}. \tag{2.6}$$

Introduce then the set of *commuting* operators $\mathcal{I}_{2k+1}$ acting on the multiparticle states, with

$$\mathcal{I}_{2k+1}|\beta_1 \dots \beta_n >_{C_1,\dots,C_n} = \frac{1}{2}\left(\sum_i e^{(2k+1)\beta_i}\right)|\beta_1 \dots \beta_n >_{C_1,\dots,C_n}, \tag{2.7}$$

where $C = L, R$ designates the chirality. One can then write

$$|B >= \exp\left[\sum_{k=0}^{\infty} \frac{-2}{2k+1}\ e^{-(2k+1)\beta_B} \mathcal{I}_{2k+1}\right]|B_{IR}\rangle. \tag{2.8}$$

Of course, the $\mathcal{I}_{2k+1}$ can be expressed in terms of the creation/annihilation operators, $\mathcal{I}_{2k+1} = \int \frac{d\beta}{4\pi} e^{(2k+1)\beta}\left[Z_L^*(\beta) Z_L(\beta) + Z_R^*(\beta) Z_R(\beta)\right]$. Using the mode expansion of the fermions, one checks this coincides with $\int_{-\infty}^{\infty} dy \mathcal{O}_{2k+2}$, where $\mathcal{O}_{2k+2}$ is defined in (2.5).

We can now write a reasonable conjecture for the hamiltonian (2.2) in the crossed channel - the reason why it is a conjecture only is because the exponential in (2.8) is determined by the action on one particular state only, $|\mathcal{B}_{IR}\rangle$, and not in true generality (one can determine the action of the exponential on other states with "momentum" actually, but still, not on all possible states of the theory). Observe now that if $\mathcal{H} = \mathcal{H}_{IR} + \delta\mathcal{H}$, the boundary state will generally read $|B\rangle = \mathcal{P}\exp\left[-\int_{-\infty}^{\infty} dy \delta\mathcal{H}\right] |B_{IR}\rangle$, where $\mathcal{P}$ is the (y) path ordered exponential. Using that the $\mathcal{I}_{2k+1}$ form a set of commuting quantities, together with the fact that the $\mathcal{O}_{2k+2}$ are self and mutually local operators, we obtain therefore

$$\mathcal{H} = \mathcal{H}_{IR} + \sum_{k=0}^{\infty} \frac{2}{2k+1} e^{-(2k+1)\beta_B} \mathcal{O}^o_{2k+2}(0), \tag{2.9}$$

again up to total derivatives.

Because the $\mathcal{I}_{2k+1}$ form a set of commuting quantities, the perturbation of the IR hamiltonian in (2.9) is, formally, integrable. This is an expected result, since after all the flow from the UV to the IR fixed point *is* integrable, a feature that should be observed from both extremities - and provides an immediate check of (2.9).

2.3. The boundary free energy

We now discuss the relation between the IR hamiltonian, and the boundary free energy.

Consider thus the theory defined for $x \in [-\infty, 0]$ and $y \in [0, 1/T]$, with periodic boundary conditions in the y direction. In the closed string point of view, the theory is thus defined on a circle instead of the infinite line, while in the open string point of view, it is now at finite temperature T.

To compute the free energy, it is convenient to "unfold" the problem, so now $x \in [-\infty, \infty]$, and the boundary interaction becomes an "impurity interaction" acting only on the R movers. Notice that a different unfolding is appropriate to study the vicinity of the UV and the IR fixed point; in one case, one extends the theory to $x > 0$ by setting $\psi_R(x, y) = \psi_L(-x, y)$, while in the other one sets of course $\psi_R(x, y) = -\psi_L(-x, y)$. In what follows, we discuss mostly the vicinity of the IR fixed point, and thus use the second folding. From the resulting "impurity" point of view, the IR fixed point is then just described by free R moving fermions.

Introducing then, in the closed string channel

$$\mathcal{I}_{2k+1} = \int_0^{1/T} \mathcal{O}_{2k+2}(z) dy = \frac{1}{2} \int_0^{1/T} dy : \psi_R \partial_z^{2k+1} \psi_R :, \tag{2.10}$$

it follows from the expression of the boundary state that [3]

$$f = -T \ln g_{IR} - T \sum_{k=0}^{\infty} \frac{-2}{2k+1} e^{-(2k+1)\beta_B} {}_{1/T}\langle 0 | \mathcal{I}_{2k+1} | 0 \rangle_{1/T}, \tag{2.11}$$

where $g_{IR} = {}_{1/T}\langle 0 | B_{IR} \rangle_{1/T}$ is the boundary degeneracy of the IR boundary state (actually independent of T [10]) and $|0\rangle_{1/T}$ denotes the ground state of the theory on a circle of circumference $1/T$.

The ground state on a circle corresponds to fermions with antiperiodic boundary conditions. Using the mode expansion of the fermions, and the expression of $\mathcal{I}_{2k+1}$ as the sum of the $2k+1^{th}$ powers of the energy, it follows that[4]

$$\begin{aligned} {}_{1/T}\langle 0 | \mathcal{I}_{2k+1} | 0 \rangle_{1/T} &= \frac{1}{2} (2\pi T)^{2k+1} \langle 0 | \sum_{j=-\infty}^{\infty} (j+1/2)^{2k+1} \psi_{-j-1/2} \psi_{j+1/2} | 0 \rangle \\ &= -\frac{1}{2} (2\pi T)^{2k+1} \sum_{j=0}^{\infty} (j+1/2)^{2k+1}. \end{aligned} \tag{2.12}$$

The sum can be evaluated by ζ-function regularization leading to

$${}_{1/T}\langle 0 | \mathcal{I}_{2k+1} | 0 \rangle_{1/T} = \frac{1}{2} (2\pi T)^{2k+1} \left(1 - \frac{1}{2^{2k+1}}\right) \zeta(-2k-1), \tag{2.13}$$

(the same computation would give a vanishing result for even powers due to $\zeta(-2k) = 0$). For $k = 0$ one gets $-\frac{\pi c T}{12}$ with $c = 1/2$; this is because $\mathcal{I}_1$ is nothing but the zero mode of the stress energy tensor [5] on a circle, $\mathcal{I}_1 = (2\pi T)\left(L_0 - \frac{c}{24}\right)$. By plugging the results (2.13) back in (2.11), one obtains an explicit expression for f.

Of course, the computation can be done in the open string channel as well; the expression (2.9) for the hamiltonian leads to

$$f = -T \ln g_{IR} + \sum_{k=0}^{\infty} \frac{2}{2k+1} e^{-(2k+1)\beta_B} \langle \mathcal{O}^o_{2k+2} \rangle_T. \tag{2.14}$$

[3] Here we have subtracted all extensive non universal terms.

[4] We use here the well known fact that the "conformal normal ordering" is related to the "operator normal ordering" by zeta regularization of the divergent parts.

[5] For the Ising model considered here, $T_{zz} = \pi : \psi_R \partial \psi_R :$

Here, $\langle \mathcal{O}^o_{2k+2}\rangle_T$ follows from expression (2.5) evaluated in the multiparticle basis of the open channel:

$$\langle \mathcal{O}^o_{2k+2}\rangle_T = (-1)^{k+1}\frac{1}{L}\int \frac{d\beta}{2\pi} e^{(2k+1)\beta}\langle Z^*_R(\beta)Z_R(\beta)\rangle_T, \tag{2.15}$$

and the notation $\langle . \rangle_T$ designates the thermal average in the theory at temperature T.

A standard thermodynamic analysis gives

$$\begin{aligned}\langle \mathcal{O}^o_{2k+2}\rangle_T &= (-1)^{k+1}\frac{1}{L}\int_{-\infty}^{\infty} d\beta e^{(2k+1)\beta}\rho(\beta)\\ &= (-1)^{k+1}\frac{1}{L}\int_{-\infty}^{\infty} d\beta e^{(2k+1)\beta}(\rho(\beta)+\tilde\rho(\beta))\frac{1}{1+\exp(\epsilon/T)}\\ &= (-1)^{k+1}\int_{-\infty}^{\infty}\frac{d\beta}{2\pi}e^{(2k+1)\beta}\frac{d\epsilon}{d\beta}\frac{1}{1+\exp(\epsilon/T)},\end{aligned} \tag{2.16}$$

where, for free fermions, $\epsilon = e^\beta$, and thus $2\pi(\rho+\tilde\rho) = L\frac{d\epsilon}{d\beta}$, a result that generalizes to interacting theories. By expanding the filling fraction, one obtains

$$\langle \mathcal{O}^o_{2k+2}\rangle_T = (-1)^{k+1}\frac{(2k+1)!}{2\pi}T^{2k+2}\left(1-\frac{1}{2^{2k+1}}\right)\zeta(2k+2). \tag{2.17}$$

To compare (2.13) and (2.17) recall the identities [11]

$$\zeta(2k+2) = \frac{(2\pi)^{2k+2}}{2(2k+2)!}(-1)^k B_{2k+2}; \qquad \zeta(-2k-1) = -\frac{B_{2k+2}}{2k+2},$$

where B_n are Bernouilli numbers. Hence, as of course should be, $\langle \mathcal{O}^o_{2k+2}\rangle_T = T_{1/T}\langle 0|\mathcal{I}_{2k+2}|0\rangle_{1/T}$, and we find the same expression for the impurity free energy [12].

Using the thermodynamic expression for the integrals of motion (the first equation of (2.16)), we obtain an alternate formula for the impurity free energy:

$$\begin{aligned} f &= -T\ln g_{IR} + \int\frac{d\beta}{2\pi}\sum_{k=0}^{\infty}\frac{2}{2k+1}\, e^{(2k+1)(\beta-\beta_B)}(-1)^{k+1}\frac{d\epsilon}{d\beta}\frac{1}{1+e^{\epsilon(\beta)/T}}\\ &= -T\ln g_{UV} - T\int\frac{d\beta}{2\pi}\frac{1}{\cosh(\beta-\beta_B)}\ln\left(1+e^{-\epsilon(\beta)/T}\right),\end{aligned} \tag{2.18}$$

where we used the fact that $f\approx -T\ln g_{UV}$ (resp. $f\approx -T\ln g_{IR}$) as $\beta_B\to-\infty$ (resp. $\beta_B\to\infty$). This last expression coincides with a well known formula obtained using the thermodynamic Bethe ansatz. It reads as well

$$f = -T\ln g_{UV} - T\int\frac{d\beta}{2\pi}\frac{1}{i}\frac{d}{d\beta}\ln R(\beta-\beta_B)\ln\left(1+e^{-\epsilon(\beta)/T}\right), \tag{2.19}$$

a result that follows directly from the form of the boundary state, and the manipulations in (2.16).

In the foregoing paragraphs, we have thus showed how the IR action could be extracted from the R matrix, and how it was closely related with the boundary free energy.

2.4. Flowing "back" from the IR fixed point.

The previous analysis shows very clearly how the TBA results are directly connected with an IR description of the flow; in fact, the free energy provides an immediate reading of the *complete IR action* (that the impurity free energy has to do with conserved quantities was observed in the earlier papers on the subject already, see [13]). It is important to realize that all this works for a particular *regularization* scheme, involving dimensional regularization and (or) contour deformation. This is somewhat obvious since the integrable approach does not involve any length scale that could act as a cut-off. The quickest way to see this more explicitely is to consider for instance the quantity $\mathcal{I}_1$. Using the mode expansion, $|0\rangle_{1/T}$ is clearly an *eigenstate* of $\mathcal{I}_1$, and thus

$$ {}_{1/T}\langle 0|\left(\mathcal{I}_1\right)^p|0\rangle_{1/T} = \left({}_{1/T}\langle 0|\mathcal{I}_1|0\rangle_{1/T}\right)^p = \left(-\frac{\pi T}{24}\right)^p . \tag{2.20} $$

To write (2.20), we have used an operator formalism, which, in fact requires "time ordering" - here ordering along x. In other words, in (2.20), the divergences have been regulated by slightly displacing the p contours of integration.

The effect of this displacement can be seen by using the fermion propagators and Wick's theorem. Aside from the term involving the (non vanishing) average of $\mathcal{I}_1$ on the circle, contractions contribute integrals with strong short distance divergences, the simplest one being

$$ \int_0^{1/T} dy_1 dy_2 \frac{1}{\left[\sin \pi T(y_1-y_2)\right]^4} . $$

If one evaluates this integral by displacing the contours and using the residue theorem (together with the periodicity of the integrand), one finds indeed a vanishing result, because the integrand has a vanishing residue at the origin.

Equivalently, in dimensional regularization, one considers the more general integral where the power is a number α ($\alpha = 4$ here) and one computes the integral in the domain of α's where it is defined. This gives

$$ \pi^2 T^2 \frac{\pi 2^{1+\alpha}\Gamma(2-\alpha)}{(1-\alpha)\Gamma^2(1-\frac{\alpha}{2})} $$

One then continues analytically to $\alpha = 4$ - and the last expression vanishes again, this time due to the double pole (in α) in the denominator. This generalizes to all the other integrals, so that the dimensionally regularized value of $\mathcal{I}_1^p$ is contributed only by its average, ie the result (2.20).

In general, integrals involving local conserved quantities (all the ones in the Ising model are of that type) will be regulated by operator methods or contour displacement, since there is no readily available parameter to perform continuations (the prescription is not ambiguous thanks to the commutativity of the conserved quantities). Integrals involving non local conserved quantities will be regulated by continuation in the parameter g. We refer to this scheme as an “analytic” regularization.

Using the previous ideas, it is clear that we can solve the problem of the most general perturbation of the IR fixed point. For an action

$$\mathcal{H} = \mathcal{H}_{IR} + \sum_{k=0}^{\infty} b_{2k+1} \mathcal{O}^o_{2k+2}, \tag{2.21}$$

the boundary free energy simply reads

$$f = -T \ln g_{IR} + \int \frac{d\beta}{2\pi} \sum_{k=0}^{\infty} b_{2k+1} e^{(2k+1)\beta} (-1)^{k+1} \frac{d\epsilon}{d\beta} \frac{1}{1 + e^{\epsilon(\beta)/T}}, \tag{2.22}$$

the integrals themshelves being evaluated in (2.16),(2.17).

As the temperature is lowered, ie when one considers this system at larger and larger scales, one simply flows to the IR fixed point, as physically expected, since all the operators $\mathcal{O}^o_{2k+2}$ are irrelevant near this fixed point. As the temperature is increased, ie when one considers the system at smaller and smaller scales, or tries to “flow back”, what happens generically is that no fixed point is reached; rather, the amplitude of all the terms becomes bigger and bigger, as expected for irrelevant perturbations. The cases where one flows back to an interesting fixed point are the ones for which the series in (2.22) defines a function of T which, continued beyond the radius of convergence, has a finite $T \to \infty$ limit. Though we do not know any definite mathematical statement about that question, it seems clear that these cases are extremely rare. For instance, the choice $b_{2k+1} = \frac{2}{2k+1} e^{-(2k+1)\beta_B}$ guarantes a flow back to the free fixed point, but any perturbation that differs, even infinitesimally, from this one by a finite number of terms, will *not* flow back to the free fixed point at all.

A quick way to build an IR hamiltonian that has a $T \to \infty$ limit is to multiply the reflection matrix by a CDD factor. By flowing backwards, one finds in this case that the difference $g_{UV} - g_{IR}$ is increased by a term $\ln \sqrt{2}$, corresponding presumably to the appearance of additional boundary degrees of freedom in the UV.

3. Approach to the IR fixed point for the spin $1/2$ Kondo model and the boundary sine-Gordon model

3.1. The Kondo model

The previous structure generalizes in a slightly more complicated form to the case of the spin 1/2 Kondo model with action

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[(\partial_x \Phi)^2 + (\Pi)^2\right] + \lambda \left[S_- e^{i\sqrt{2\pi g}\Phi(0)} + S_+ e^{-i\sqrt{2\pi g}\Phi(0)}\right], \tag{3.1}$$

where S is a spin one half operator (λ is assumed positive in what follows). The boundary interaction is integrable, and the same manipulations we carried out for the Ising model can be acomplished here too.

Instead of describing the bulk with massless fermions, we use massless L and R moving solitons and antisolitons. Parametrizing their energy by a rapidity $e = \pm p = e^{\beta}$, these particles have factorized scattering, the LL and RR scattering being given by an S matrix which, as a function of the rapidities, is the same as the S matrix of the bulk sine-Gordon model, $S_{LL} = S_{RR} = S^{SG}$, while the LR scattering is trivial. The solitons and antisolitons scatter off the boundary one by one with no particle production, and the R matrix is given by

$$\begin{aligned} R_{\pm}^{\mp} &\equiv R = -i \tanh\left(\frac{\beta - \beta_B}{2} - \frac{i\pi}{4}\right) \\ R_{\pm}^{\pm} &= 0. \end{aligned} \tag{3.2}$$

In the so called repulsive regime, that is for $g \geq \frac{1}{2}$, there are no bound states, and the soliton and antisoliton are the only particles in the spectrum. The boundary state can be written in a form similar to (2.4)

$$\begin{aligned} |B\rangle \propto \sum_{n=0}^{\infty} \int_{\beta_1 < \ldots < \beta_n} \prod_i \frac{d\beta_i}{2\pi} K(\beta_i - \beta_B) \sum_{\epsilon_i = \pm} Z^*_{L,\epsilon_1}(\beta_1) \ldots Z^*_{L,\epsilon_n}(\beta_n) \\ \times Z^*_{R,\epsilon_1}(\beta_1) \ldots Z^*_{R,\epsilon_n}(\beta_n)|0\rangle, \end{aligned} \tag{3.3}$$

with, as for the Ising case, $K(\beta - \beta_B) = i \tanh \frac{\beta_B - \beta}{2}$. As a result, introducing (C denotes the chirality, $C = L, R$)

$$\mathcal{I}_{2k+1}|\beta_1 \ldots \beta_n\rangle_{C_1,\epsilon_1 \ldots} = \frac{\lambda_{2k+1}}{2}\left(\sum_i e^{(2k+1)\beta_i}\right)|\beta_1 \ldots \beta_n\rangle_{C_1,\epsilon_1 \ldots}, \tag{3.4}$$

we can write

$$|B\rangle = \exp\left[\sum_{k=0}^{\infty} \frac{-2}{(2k+1)} e^{-(2k+1)\beta_B} \frac{\mathcal{I}_{2k+1}}{\lambda_{2k+1}}\right] |B_{IR}\rangle. \tag{3.5}$$

The coefficients λ_{2k+1} in (3.4) will be adjusted for later convenience.

Indeed, a new difficulty arises here when one wishes to reexpress the set of commuting quantities $\mathcal{I}_{2k+1}$ in terms of local fields. As far as we know, this question was first addressed quantitatively in [12], where the first few conserved quantities were studied numerically using the TBA. The following analytical expression was obtained in unpublished works by the present authors, as well as by Al. Zamolodchikov [14], and probably by a few others too. A derivation is presented in the appendix for completeness; to our knowledge, it has never appeared elsewhere, though the technique is hardly original.

To proceed, we need to chose some normalizations. We first introduce the twisted stress energy tensor

$$T_{zz} = -2\pi : (\partial\phi)^2 : + i(1-g)\sqrt{\frac{2\pi}{g}}\partial^2\phi, \tag{3.6}$$

where $\phi \equiv \phi_R$ is the right moving component of the boson. The central charge corresponding to this tensor is

$$c = 1 - 6\frac{(1-g)^2}{g}. \tag{3.7}$$

A set of commuting quantities is then obtained by integrating successive powers of this stress energy tensor. We define (the 2π normalization makes subsequent formulas simpler)

$$\begin{aligned}
\mathcal{O}_2 &= \frac{1}{4\pi}\left(T_{zz} + T_{\bar{z}\bar{z}}\right) \\
\mathcal{O}_4 &= \frac{1}{4\pi}\left(: T_{zz}^2 + R \to L\right) \\
\mathcal{O}_6 &= \frac{1}{4\pi}\left(: T_{zz}^3 : -\frac{c+2}{12} : T_{zz}\partial^2 T_{zz} : + R \to L\right) \\
&\dots
\end{aligned} \tag{3.8}$$

The normalization is such that $\mathcal{O}_{2k+2}$ goes as $\frac{1}{2}(-1)^{k+1}(2\pi)^k(\partial\phi)^{2k+2}$. We then define $\mathcal{I}_{2k+1} = \int_{-\infty}^{\infty} \mathcal{O}_{2k+2} dy$. Let us stress that these quantities commute at the conformal point only (in fact, of course, their left and right components independently commute). In the massive sine-Gordon model (ie with the bulk perturbation $\cos 2\sqrt{2\pi g}\Phi$ in our notations), there exists non chiral deformations of these quantities that still commute, and act as sums of odd powers of momenta on the (massive) multiparticle states [15]. In the massless scattering description we are using here, one considers the free boson as the limit of the

massive sine-Gordon model, and the particular chiral quantities $\mathcal{I}_{2k+1}$ are singled out, which act again as sums of odd powers of momenta on the multiparticle states. Of course, there are more conserved quantities right at the conformal point, but they do not seem to have any simple meaning in terms of rapidities - see next section however, and [12] for more details.

With this choice, one has (see the appendix)

$$\lambda_{2k+1} = \left(\frac{\pi}{g}\right)^k (k+1)! \frac{\Gamma\left[\frac{(2k+1)g}{2(1-g)}\right]}{\left(\Gamma\left[\frac{g}{2(1-g)}\right]\right)^{2k+1}} \frac{\left(\Gamma\left[\frac{1}{2(1-g)}\right]\right)^{2k+1}}{\Gamma\left[\frac{(2k+1)}{2(1-g)}\right]}. \tag{3.9}$$

In the following, we will also need the relation between the parameter β_B of the R matrix and the coupling λ in the action of the Kondo model. This was determined in [3], and reads

$$T_B = \frac{\Gamma\left(\frac{g}{2(1-g)}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2(1-g)}\right)} \left[\lambda\Gamma(1-g)\right]^{1/(1-g)}. \tag{3.10}$$

From (3.5), we then obtain

$$\mathcal{H} = \mathcal{H}_{IR} + \sum_{k=0}^{\infty} b_{2k+1} \lambda^{-\frac{1+2k}{1-g}} \mathcal{O}^o_{2k+2}, \tag{3.11}$$

with

$$\begin{aligned} b_{2k+1} =& \frac{2}{2k+1} e^{-(2k+1)\beta_B} \frac{\lambda^{\frac{1+2k}{1-g}}}{\lambda_{2k+1}} \\ =& \sqrt{\pi} \frac{g^{k+1}}{(1-g)(k+1)!} \frac{\Gamma\left[(k+1/2)\frac{1}{1-g}\right]}{\Gamma\left[1+(k+1/2)\frac{g}{1-g}\right]} \left[\Gamma(1-g)\right]^{-\frac{1+2k}{1-g}}, \end{aligned} \tag{3.12}$$

and $\mathcal{O}^o_{2k+2}$ follows from the expression for $\mathcal{O}_{2k+2}$ by replacing z by w, and multiplying by an overall factor $(-1)^{k+1}$:

$$\begin{aligned} \mathcal{O}^o_2 &= -\frac{1}{4\pi} \left(T_{ww} + T_{\bar{w}\bar{w}}\right) \\ \mathcal{O}^o_4 &= \frac{1}{4\pi} \left(: T^2_{ww} + R \to L\right) \\ \mathcal{O}^o_6 &= -\frac{1}{4\pi} \left(: T^3_{ww} : - \frac{c+2}{12} : T_{ww} \partial^2 T_{ww} : + R \to L\right) \\ &\dots \end{aligned} \tag{3.13}$$

The foregoing results essentially coincide with those in [5]. Our route is quite different however; in particular, the form of the R matrix or the normalization of the integrals of motion are not used at all in [5], where, instead, a functional relation approach is developed.

The Kondo model is a very interesting physical example from the point of view of the IR perturbation theory. For any value of g (which physically corresponds to the anisotropy), the IR fixed point is always the same (see [16] for details and references). To get back to a g dependent UV fixed point, one needs to perturb the IR fixed point by the same family of operators (stress tensor and the like) but with coefficients that depend on g: it is only through this fine tuning of the coefficients that different flows can be obtained. For a given g, the free energy for an arbitrary IR perturbation - that expands on the conserved quantities - has an expression similar to what we wrote in the Ising model.

The foregoing analysis could be generalized to the regime where the associated bulk sine-Gordon model has bound states, that is $g < \frac{1}{2}$. The final expressions involving quantum fields, for instance (3.11), would *not* change; they are expected to be analytical in g, a result that can easily be checked using the method we explain below. On the other hand, expressions involving scattering quantities *would* change . Here, we would like to make a remark concerning (3.4). Because in the scattering the numbers of solitons and breathers are independently conserved, one expects in general a result of the form, introducing the color ϵ_i for particles ($\epsilon = 1, \ldots, m, \ldots$ for breathers, $\epsilon = \pm 1$ for solitons antisolitons)

$$\mathcal{I}_{2k+1}|\beta_1, \ldots, \beta_n\rangle_{C_1,\epsilon_1,\ldots} = \frac{1}{2}\left(\sum_i \lambda_{2k+1,\alpha_i} e^{(2k+1)\beta_i}\right)|\beta_1, \ldots, \beta_n\rangle_{C_1,\epsilon_1,\ldots}, \tag{3.14}$$

where the $\lambda_{2k+1,\alpha}$ are a priori all different. The determination of these factors is an interesting problem by itself. It can be quickly solved if one observes that the formula for the boundary state (3.3) immediately generalizes to the case where breathers are present in the spectrum, by using the m-breather reflection matrix

$$R_m = -\frac{\tanh\left(\frac{\beta-\beta_B}{2} - \frac{i\pi}{4}\frac{mg}{1-g}\right)}{\tanh\left(\frac{\beta-\beta_B}{2} + \frac{i\pi}{4}\frac{mg}{1-g}\right)}. \tag{3.15}$$

Expanding $\frac{1}{i}\frac{d}{d\beta}\ln R$ in (odd) powers of e^{β}, one finds

$$\begin{aligned} \frac{1}{i}\frac{d}{d\beta}\ln R_m &= 4\sum_{k=0}^{\infty} e^{-(2k+1)\beta}\sin\left[m\pi\frac{(2k+1)g}{2(1-g)}\right] \\ \frac{1}{i}\frac{d}{d\beta}\ln R &= 2\sum_{k=0}^{\infty}(-1)^k e^{-(2k+1)\beta}. \end{aligned} \tag{3.16}$$

By putting these expansions in the formula for the boundary state, it follows that the ratio of normalizations of conserved quantities is the same as the ratio of the odd powers of e^β in (3.16), that is

$$\frac{\lambda_{2k+1,m}}{\lambda_{2k+1,\pm}} = 2(-1)^k \sin\left[m\pi\frac{(2k+1)g}{2(1-g)}\right]. \tag{3.17}$$

The rest of the arguments follows with minor modifications.

The key feature of the spin 1/2 Kondo problem is that the IR fixed point is approached along the conserved quantities $\mathcal{O}^o_{2k+2}$ of even dimensions. The situation is more interesting for the higher spin case, or the boundary sine-Gordon case.

3.2. The boundary sine-Gordon problem

The previous structure generalizes in a slightly more complicated form to the boundary sine-gordon model

$$\mathcal{H} = \frac{1}{2}\int_{-\infty}^{0} dx \int_{-\infty}^{\infty} dy \left[(\partial_x\Phi)^2 + (\Pi)^2\right] + 2\lambda\cos\sqrt{2\pi g}\Phi(0). \tag{3.18}$$

The boundary interaction is integrable, and the same manipulations we carried out for the Ising model can be acomplished here too. While technically more involved, the general spirit is very similar, so we will restrict ourselves to the salient features.

The quickest way to proceed is to restrict to the attractive case of the associated bulk sine-Gordon model, $g = 1/\text{integer}$, and to consider the reflection matrices in that case [17]:

$$\begin{aligned}
\frac{1}{i}\frac{d}{d\beta}\ln R_m &= 2\sum_{k=0}^{\infty}(-1)^k e^{-(2k+1)\beta}\frac{\sin\left[m\pi(2k+1)g/2(1-g)\right]}{\sin\left[\pi(2k+1)g/2(1-g)\right]} \\
\frac{1}{i}\frac{d}{d\beta}\ln(R_+^+ \pm R_-^+) &= -\frac{1-g}{g}\sum_{k=1}^{\infty}(-1)^k e^{-2k\beta(1-g)/g}\tan k\pi\frac{1-g}{g} \\
\pm\frac{1-g}{g}\sum_{k=0}^{\infty}&(-1)^{k+1}e^{-(2k+1)\beta(1-g)/g} + \sum_{k=0}^{\infty}\frac{e^{-(2k+1)\beta}}{\sin\left[\pi(2k+1)g/2(1-g)\right]}.
\end{aligned} \tag{3.19}$$

The boundary scattering is non diagonal in the soliton antisoliton basis, but it is diagonal for the symmetric and antisymmetric combinations, which scatter with the amplitudes $R_+^+ \pm R_-^+ \equiv R_\pm$. The bulk scattering is diagonal in either basis.

We also recall that the relation between the coupling constant λ in the action and the rapidity β_B is modified in the case of the boundary sine-Gordon model, reading then

$$T_B = (2\sin\pi g)^{1/(1-g)}\frac{\Gamma\left(\frac{g}{2(1-g)}\right)}{\sqrt{\pi}\Gamma\left(\frac{1}{2(1-g)}\right)}\left[\lambda\Gamma(1-g)\right]^{1/(1-g)}. \tag{3.20}$$

By using the reflection matrices, and following the same logic as before, we obtain immediately the coefficients of all conserved quantities for the hamiltonian near the IR fixed point. We have now an expansion similar to (3.11), but with the coefficents b_{2k+1} replaced by

$$c_{2k+1} = \frac{(-1)^k}{2} \frac{1}{\sin\left[\pi(2k+1)g/2(1-g)\right]} \frac{1}{(2\sin\pi g)^{\frac{2k+1}{2(1-g)}}} b_{2k+1}, \tag{3.21}$$

where the prefactor is just the ratio of the coefficients of odd powers of $e^{-\beta}$ in (3.19) and (3.16) (of course, this ratio is the same for the breathers and the soliton antisoliton R matrices), plus an additional power of $2\sin\pi g$ arising from the difference between (3.20) and (3.10).

It is well known [18] that, at the conformal point, the (chiral part) of the quantities $\mathcal{I}_{2k+1}$ commute not only together and with the integral of the perturbation $\int_{-\infty}^{\infty} dy e^{\pm i\sqrt{8\pi g}\phi}$, but they also commute with the "dual" of the perturbation $\int_{-\infty}^{\infty} e^{\pm i\sqrt{\frac{8\pi}{g}}\phi}$. When the perturbation is turned on, a deformation of these quantities turns out to still be conserved, guaranteeing the integrability of the flow. This conservation is true all the way to the IR fixed point, where again the purely chiral quantities are conserved, by conformal invariance. It follows that, if one investigates the conservation perturbatively near the IR fixed point within the dimensionnally regularized scheme, the only operator that can be added to $\mathcal{H}_{IR}$, besides the $\mathcal{O}_{2k+2}$, is $\cos\sqrt{\frac{2\pi}{g}}\tilde{\Phi}$. Here, $\tilde{\Phi}$ is the dual of the field Φ, $\tilde{\Phi}=\phi_R-\phi_L$, and we used that $\phi_R=-\phi_L$ at the IR fixed point. By dimensional analysis, its amplitude goes as $\lambda^{-1/g}$. The exact amplitude follows from eqn (6.18) and (6.20) in [3]. One can thus finally write

$$\mathcal{H} = \mathcal{H}_{IR} + 2\lambda_d \cos\sqrt{\frac{2\pi}{g}}\tilde{\Phi} + \sum_{k=0}^{\infty} c_{2k+1}\lambda^{-\frac{1+2k}{1-g}}\mathcal{O}^o_{2k+2}, \tag{3.22}$$

where

$$\lambda_d = \frac{1}{2\pi g}\Gamma\left(\frac{1}{g}\right)\left[\frac{g\Gamma(g)}{2\pi}\right]^{\frac{1}{g}}\lambda^{-\frac{1}{g}}. \tag{3.23}$$

Observe that the R matrix elements for breathers expand only on odd powers of e^{β} - this indicates that the non local conserved quantities formed with $\cos\sqrt{\frac{2\pi}{g}}\tilde{\Phi}$ have vanishing eigenvalue on the breather states, a result of their charge neutrality.

3.3. The Kondo model with higher spin

It is necessary to still generalize the previous arguments slightly, to take into account the Kondo model with higher spin. The structure is actually very similar to the spin 1/2 Kondo and the BSG case.

The UV Kondo hamiltonian reads as (3.18) with now the spin in a spin j representation of $U_q sl(2)$, $q = e^{i\pi g}$. One finds that the hamiltonian near the IR fixed point reads

$$\mathcal{H} = \mathcal{H}_{IR} + \lambda_d \left[S_- e^{i\sqrt{\frac{2\pi}{g}}\tilde{\Phi}(0)} + S_+ e^{-i\sqrt{\frac{2\pi}{g}}\tilde{\Phi}(0)} \right] + \sum_{k=0}^{\infty} d_{2k+1} \lambda^{-\frac{1+2k}{1-g}} \mathcal{O}^o_{2k+2}, \tag{3.24}$$

where s is in the representation $j - \frac{1}{2}$, $\lambda_d \propto \lambda^{-1/g}$, and the coefficients d_{2k+1} could be determined using the same method as before (see section 5 for more details) .

4. An application: the resistivity in the Kondo model.

A good testing ground for the previous considerations is the isotropic Kondo model, where the strong coupling behaviour can be probed by experiments at low temperatures. The most interesting quantity in that case is of course the resistivity, for which no closed form results were available so far, besides the T^2 term that follows from Fermi liquid theory [19] (attempts to compute ρ with the Bethe ansatz have failed, partly because it is truly a three dimensional quantity). The method we have developed in this paper allows us to make an important progress on that question: short of getting ρ in closed form, we can at least compute it perturbatively near the strong coupling fixed point, now that we know the exact structure of the hamiltonian. This allows us to go beyond the Fermi liquid approximation, and evaluate ρ as a power series in T^2 at low temperatures.

In order to study the resistivity, we first need to go back to the 3d formulation of the system with electronic anihilation operator

$$\Psi(\vec{r}) = \int \frac{d^3\vec{p}}{(2\pi)^{3/2}} e^{i\vec{p}\cdot\vec{r}} \Psi(\vec{p}), \tag{4.1}$$

where we suppressed the spin indice for simplicity. As usual, since the Kondo interaction, when the impurities are dilute, is assumed to be with only one impurity, we can consider only the s-wave component of that operator around the Fermi points

$$\Psi(\vec{r}) = \frac{1}{2\sqrt{2}i\pi r} [e^{ik_F r}\psi_R(r) - e^{-ik_F r}\psi_L(r)] \tag{4.2}$$

with $r > 0$ and we used right and left one dimensional moving field. This decomposition implies $\psi_L(0) = \psi_R(0)$. In the interacting theory, only the s-wave parts of the three dimensional Green's function will be affected, moreover, only the LR and RL components of the dimensionally reduced model are affected by the interaction. This leads to the following form for the three dimensional interacting Green's function (for the spin up field for example)

$$\begin{aligned} G(\omega_M, &\vec{r}_1, \vec{r}_2) - G^0(\omega_M, \vec{r}_1 - \vec{r}_2) \\ &= \frac{-1}{8\pi^2 r_1 r_2}\Big[e^{-ik_F(r_1+r_2)}(G_{LR}(\omega_M, r_1, r_2) - G^0_{LR}(\omega_M, r_1, r_2)) \\ &+ e^{ik_F(r_1+r_2)}(G_{RL}(\omega_M, r_1, r_2) - G^0_{RL}(\omega_M, r_1, r_2))\Big], \end{aligned} \tag{4.3}$$

with the superscript 0 denoting the free Green function. As we will see, this is the quantity we need to compute the resistivity. The interacting LR (resp. RL) Green's functions are defined by

$$G_{C_1C_2}(\omega_M, r) = -\int_{-\beta/2}^{\beta/2} dy\ e^{i\omega_M y}\langle\psi_{C_1}(r, y)\psi_{C_2}(0, 0)\rangle, \tag{4.4}$$

with C_i indicating the chirality. As an example, we have[6]

$$\begin{aligned} G^0_{RL}(\omega_M, r) &= \int_{-\beta/2}^{\beta/2} dy \frac{e^{i\omega_M y}}{\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(-y + ir)} \\ &= -2\pi i e^{-\omega_M r}\theta(\omega_M), \end{aligned} \tag{4.5}$$

where we have used the fact that $r > 0$ and that in the UV

$$\langle\psi_R(w_1)\psi_L(\bar{w}_2)\rangle = -\frac{1}{\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w_1 - \bar{w}_2)}. \tag{4.6}$$

In the IR, the only difference is the boundary condition which will result in a change of sign in the propagator. When we put everything back into the three dimensional expression for the Green function, we get (at the IR fixed point)

$$\begin{aligned} &G^{IR}(\omega_M, \vec{r}_1, \vec{r}_2) - G^0(\omega_M, \vec{r}_1 - \vec{r}_2) \\ &= \frac{i}{2\pi r_1 r_2}\Big[e^{-ik_F(r_1+r_2)}e^{\omega_M(r_1+r_2)}\theta(-\omega_M) - e^{ik_F(r_1+r_2)}e^{-\omega_M(r_1+r_2)}\theta(\omega_M)\Big] \\ &= G^0(\omega_M, \vec{r}_1)T(\omega_M)G^0(\omega_M, -\vec{r}_2). \end{aligned} \tag{4.7}$$

[6] The fermions operators have an extra $\sqrt{2\pi}$ in their normalisation here.

Following the arguments of [20], for a dilute array of impurities of densities n_i the lowest order correction to the complete Green function takes the form

$$G(\omega_M, \vec{r}_1, \vec{r}_2) - G^0(\omega_M, \vec{r}_1 - \vec{r}_2) \simeq n_i \int d^3\vec{r}_i G^0(\omega_M, \vec{r}_1 - \vec{r}_i) T(\omega_M) G^0(\omega_M, \vec{r}_i - \vec{r}_2) \quad (4.8)$$

Summing over multi-impurity terms, the self-energy takes the simple form

$$\Sigma(\omega_M) = n_i T(\omega_M) \quad (4.9)$$

where higher orders in n_i are neglected. The retarded self-energy is found by the analytical continuation $i\omega_M \to \omega + i\eta$ leading to

$$\Sigma^R(\omega) = -\frac{in_i}{\pi\nu} \quad (4.10)$$

ν is the number of spin per channel, we have reestablished its dependance at the end since it only amounts to a factor of two (separate spins contribute the same). This is the expected result at the IR fixed point for the one channel Kondo model. Finally the resistivity follows from the Kubo formula for the conductivity

$$\frac{1}{\rho(T)} = \sigma(T) = 2\frac{e^2}{3m^2} \int \frac{d^3\vec{p}}{(2\pi)^3} \left[-\frac{dn}{d\epsilon_k}\right] \vec{p}\cdot\vec{p}\,\tau(\epsilon_k), \quad (4.11)$$

with the single particle lifetime defined by $1/\tau = -2Im\Sigma^R$. The dispersion relation $\epsilon_k = v_F k$ has been linearised in that limit.

All this discussion was done using the fermions but to continue and understand how to get away from the IR fixed point, we need to use our earlier results. To make contact with our previous discussion of the Kondo model, we need to bosonise the system. This is done using the rules

$$\psi_{L/R,\mu}(r,y) \propto e^{\pm i\sqrt{4\pi}\phi_{L/R,\mu}(r,y)}. \quad (4.12)$$

Notice that we have reestablished the spin dependence, $\mu =\uparrow, \downarrow$, since this will be crucial in the following. At the UV fixed point we have $\psi_{L,\mu}(0) = \psi_{R,\mu}(0)$ but since we are interested rather in perturbation around the IR fixed point, we impose the conditions $\psi_{R,\mu}(0) = -\psi_{L,\mu}(0)$ for the IR correlators. This leads to the RL (LR) bosonic propagator

$$\langle \phi_{R,\mu}(w_1)\phi_{L,\nu}(\bar{w}_2)\rangle = \delta_{\mu\nu}\left[-\frac{1}{4\pi}\ln\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w_1 - \bar{w}_2)\right] \quad (4.13)$$

which translates in the correct fermionic propagator when using the bosonisation rules given above. Although we are interested in computing the Green function of spin up

fields, for example, the integrable description is rather in terms of the spin and charge densities, ie introduce

$$\begin{aligned}\phi_s &= \frac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow)\\ \phi_c &= \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow).\end{aligned} \tag{4.14}$$

In terms of these fields, the interaction at the boundary only involve the spin field and is given by the hamiltonian written in the previous section. The charge field remains non-interacting. The perturbation around the IR fixed point is described by the hamiltonian

$$\mathcal{H} = \mathcal{H}_{IR} + \sum_{k=0}^{\infty} b_{2k+1}\lambda^{-\frac{1+2k}{1-g}}\mathcal{O}^o_{2k+2} \tag{4.15}$$

where all the couplings and operators have been given in section 3; the boson field in the latter section coincides with ϕ_s here. On the other hand, if we look at the bosonisation of the Green function for the spin up field, we observe that there will be contributions for each field

$$\psi_{L\uparrow} \propto e^{i\sqrt{4\pi}\phi_{L\uparrow}} = e^{i\sqrt{2\pi}(\phi_{Lc}+\phi_{Ls})} \tag{4.16}$$

and when computing the interacting left-right Green function for example, the charge sector will be completely decoupled, ie

$$\langle\cdots\rangle = \langle\cdots\rangle_{charge} \times \langle\cdots\rangle_{spin}$$

Only when doing the Fourier transform will the charge part contribute. Let us proceed to the computation to show this more explicitly.

The isotropic case ($g = 1$) leads to some simplifications in the previous expressions, leading to the identification

$$b_{2k+1}\lambda^{-\frac{(1+2k)}{(1-g)}} = \frac{1}{\pi^k(k+\frac{1}{2})(k+1)!}T_B^{-(1+2k)}. \tag{4.17}$$

The coupling T_B now is identified with the usual Kondo temperature T_K (up to a normalization that is a matter of convention, and will be decided later) and the contribution of each operator is determined through these relations. This provides the information necessary to compute higher corrections to the resistivity from the IR fixed point.

Up to order T_B^{-2} the contributions are exactly the same as the ones found previously since only one operator, the energy momentum tensor, appears to that order. It is at the

third order that the non-trivial approach to the fixed point will be needed since the second operator $\mathcal{O}_4$ will be involved. First let us proceed to reproduce results found before for the two first orders using our bosonised formulation. To first order, the leading irrelevant operator is (with the proper normalisation)

$$
\begin{aligned}
-\frac{1}{4\pi}(T_{ww}+T_{\bar{w}\bar{w}}) &= \frac{1}{2}[:(\partial_w\phi_s)^2:+:(\partial_{\bar{w}}\phi_s)^2:] - \frac{(2\pi T^2)}{24} \\
&=: (\partial_y\phi_s)^2 : - \frac{(2\pi T^2)}{24}
\end{aligned}
\tag{4.18}
$$

where we have used the fact that the operator is inserted at $r=0$ to get the last line. The constant is a disconnected term that gets cancelled when dividing by the partition function to evaluate the correlator: we can thus forget about it in what follows. Inserting (4.18) in the correlator (of the relevant RL or LR components) we get the lowest order contribution to the one dimensional propagators (again for the spin up field for example)

$$
\begin{aligned}
\frac{2}{T_B}\int_{-\beta/2}^{\beta/2} dy dy'\ e^{i\omega_M y}\langle e^{\pm i\sqrt{2\pi}[\phi_{R/L,c}+\phi_{R/L,s}](r_1,y)} : (\partial_{y'}\phi_s)^2 : \times \\
\times e^{\mp i\sqrt{2\pi}[\phi_{L/R,c}+\phi_{L/R,s}](r_2,0)}\rangle_{IR}
\end{aligned}
\tag{4.19}
$$

with the subscript IR meaning that we evaluate the propagators with respect to the IR action. Note that the contribution from the charge boson decouples and the perturbation only affects the spin sector. Again let us write explicitely the RL component: we have for the first correction

$$
\begin{aligned}
\delta^{(1)}G_{RL} &= \frac{-1}{4\pi T_B}\int_{-\beta/2}^{\beta/2} dy dy'\ e^{i\omega_M y}\frac{\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w_1-\bar{w}_2)}{[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w'-\bar{w}_2)]^2[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w'-w_1)]^2} \\
&= -\frac{2i\pi}{T_B}\epsilon(\omega_M)\omega_M e^{-\omega_M(r_1+r_2)},
\end{aligned}
\tag{4.20}
$$

with $\epsilon(\omega_M)$ the step function. This leads to a correction of the self-energy of the form

$$
\Sigma^R(\omega) = -\frac{in_i}{2\pi\nu}\left[2+i\frac{\omega}{T_B}\right]
\tag{4.21}
$$

which is the expected form. The correction is real and does not contribute to the conductivity or the life time. To get bona-fide contributions, we need to go further in the IR perturbation theory. To next order, the conserved quantity $\mathcal{O}_2$ will contribute again but the higher quantity $\mathcal{O}_4$ will not yet give a contribution. So to second order, we have

$$
\begin{aligned}
-\frac{2}{T_B^2}\int_{-\beta/2}^{\beta/2} dy dy' dy''\ \frac{e^{i\omega_M y}}{[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w_1-\bar{w}_2)]^{1/2}}\times \\
\times\langle e^{i\sqrt{2\pi}\phi_{R,s}(w_1)} : (\partial_{y'}\phi_s)^2 :: (\partial_{y''}\phi_s)^2 : e^{-i\sqrt{2\pi}\phi_{L,s}(\bar{w}_2)}\rangle_{IR}
\end{aligned}
\tag{4.22}
$$

where we already contracted the charge part. Using the relation

$$\begin{aligned}
&:(\partial_{y'}\phi(0,y'))^2::(\partial_{y''}\phi(0,y''))^2:=:(\partial_{y'}\phi(0,y'))^2(\partial_{y''}\phi(0,y''))^2:\\
&+4\left(\frac{-1}{4\pi[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(y'-y'')]^2}\right):(\partial_{y'}\phi(0,y'))(\partial_{y''}\phi(0,y'')):\\
&+2\left(\frac{-1}{4\pi[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(y'-y'')]^2}\right)^2,
\end{aligned}\tag{4.23}$$

we get three contributions to the second order, two of which are divergent. The regularisation of divergences here is done by analyticity, as explained in section 3: we slightly modify the contours of the y'' integral, and move it by $i\delta$ in the complex plane. The integrals are then done by simple residue evaluation. Usually there could be a dependance on the way the contour is deformed but this disapears here since the operators commute with each other (there is no simple pole in their OPE). The last term in the expansion has no frequency dependence and the explicit evaluation (using our prescription for the regularisation of the divergence) gives zero. The first contribution has the form

$$\begin{aligned}
\delta^{(2a)}G_{RL}=&-\frac{2}{T_B^2(8\pi)^2}\int_{-\beta/2}^{\beta/2}dydy'dy''e^{i\omega_M y}\ [\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w_1-\bar{w}_2)]^3\times\\
&\times\left\{\frac{1}{[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w'-w_1)][\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w'-\bar{w}_2)]}\right\}^2\times\\
&\times\left\{\frac{1}{[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w''-w_1)][\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w''-\bar{w}_2)]}\right\}^2
\end{aligned}\tag{4.24}$$

which contains no divergences and can be evaluated straightforwardly by the method of residues. The integral over y', y'' leads to

$$\delta^{(2a)}G_{RL}=\frac{1}{2T_B^2}\int_{-\beta/2}^{\beta/2}dye^{i\omega_M y}\frac{[\cos\frac{\pi}{\beta}(w_1-\bar{w}_2)]^2}{[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w_1-\bar{w}_2)]^3},\tag{4.25}$$

and evaluation of the integral gives

$$\delta^{(2a)}G_{RL}=\frac{i\pi}{2T_B^2}\epsilon(\omega_M)e^{-\omega_M(r_1+r_2)}[\omega_M^2+(\pi T)^2].\tag{4.26}$$

The second contribution, which has divergences, takes the form

$$\begin{aligned}
\delta^{(2b)}G_{RL}=&\frac{-1}{4\pi^2T_B^2}\int_{-\beta/2}^{\beta/2}dydy'dy''e^{i\omega_M y}\ \frac{1}{[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(y'-y'')]^2}\times\\
&\times\frac{\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w_1-\bar{w}_2)}{[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w'-\bar{w}_2)][\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w'-w_1)][\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w''-\bar{w}_2)][\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w''-w_1)]}
\end{aligned}\tag{4.27}$$

and evaluating by residues leads to

$$\delta^{(2b)} G_{RL} = \frac{i\pi}{2T_B^2}\epsilon(\omega_M)e^{-\omega_M(r_1+r_2)}(2\omega_M^2 - 2(\pi T)^2). \tag{4.28}$$

So the total contribution to second order to the retarded green's function takes the form (once we analytically continue to real frequencies)

$$\Sigma^R(\omega) == -\frac{in_i}{2\pi\nu}\left[2 + i\frac{\omega}{T_B} - \frac{1}{4T_B^2}(3\omega^2 + (\pi T)^2)\right] \tag{4.29}$$

and as expected we have a universal function of $(\omega/T_K, T/T_K)$. The previous results did not require any information about the other operators but at third order, the operator

$$\begin{aligned} \mathcal{O}_4 &= -\frac{1}{4\pi}[: T_{ww}^2 : + : T_{\bar{w}\bar{w}}^2 :] \\ &= \pi\left[: (\partial_y\phi_s)^4 : - \frac{1}{2\pi} : \partial_y\phi_s\partial_y^3\phi_s :\right] - \frac{(\pi T)^2}{2} : (\partial_y\phi_s)^2 : + \frac{3(\pi T)^4}{80\pi} \end{aligned} \tag{4.30}$$

needs to be taken into accout: it comes with the coupling $1/(3\pi T_B^3)$ in the hamiltonian. Using the relation

$$\begin{aligned} : (\partial_{y'}\phi)^n : : e^{-i\sqrt{2\pi}\phi_L(\bar{w}_2)} :=& \sum_{p=0}^{n}\binom{n}{p}\left(\frac{i}{\sqrt{8\pi}[\frac{\beta}{\pi}\sin\frac{\pi}{\beta}(w' - \bar{w}_2)]}\right)^{n-p} \times \\ & \times : (\partial_{y'}\phi)^p e^{-i\sqrt{2\pi}\phi_L(\bar{w}_2)} : \end{aligned} \tag{4.31}$$

we get, using the residue theorem, the contribution,

$$\delta^{(3a)} G_{RL} = \frac{1}{24T_B^3}\epsilon(\omega_M)e^{-\omega_M(r_1+r_2)}[6(i\omega_M)^3 + 6i\omega_M(\pi T)^2]. \tag{4.32}$$

There is also a contribution from the leading irrelevant operator when expanded to third order, which reads

$$\delta^{(3b)} G_{RL} = \frac{\pi}{6T_B^3}\epsilon(\omega_M)e^{-\omega_M(r_1+r_2)}[3i\omega_M(\pi T)^2 + 5(i\omega_M)^3] \tag{4.33}$$

At this order, the contributions are all imaginary and we need to go the the next order to get non trivial contributions to the resistivity. At fourth order there are two contributions, one coming from the leading operator only, $\mathcal{O}_2^4$, and another, from the combination of the leading and next to leading operators, $\mathcal{O}_2\mathcal{O}_4$. The computation are analogous to the

previous ones, but more tedious. The final result for the retarded self energy up to fourth order is

$$\begin{aligned}\Sigma^R(\omega) = &-\frac{in_i}{2\pi\nu}\left[2 + i\frac{\omega}{T_B} - \frac{1}{4T_B^2}\left(3\omega^2 + (\pi T)^2\right) - \right.\\ &- i\left(\frac{5}{12} + \frac{3}{24\pi}\right)\left(\frac{\omega}{T_B}\right)^3 - i\left(\frac{1}{4} + \frac{1}{8\pi}\right)\frac{\omega}{T_B}\left(\frac{\pi T}{T_B}\right)^2 + \\ &+ \left(\frac{35}{192} + \frac{7}{32\pi}\right)\left(\frac{\omega}{T_B}\right)^4 + \left(\frac{19}{96} + \frac{5}{16\pi}\right)\left(\frac{\pi T}{T_B}\right)^2\left(\frac{\omega}{T_B}\right)^2 + \\ &\left. + \left(\frac{11}{192} + \frac{3}{32\pi}\right)\left(\frac{\pi T}{T_B}\right)^4\right]\end{aligned} \tag{4.34}$$

Using the Kubo formula this leads to our main result for the resistivity (which we computed to sixth order)

$$\begin{aligned}\rho(T) = &\frac{3n_i}{(\pi v_F \nu)^2}\left[1 - \frac{1}{4}\left(\frac{\pi T}{T_B}\right)^2 + \left(\frac{13}{240} + \frac{3}{20\pi}\right)\left(\frac{\pi T}{T_B}\right)^4\right.\\ &\left. + \left(\frac{47}{10080} - \frac{1}{8\pi} - \frac{53}{336\pi^2}\right)\left(\frac{\pi T}{T_B}\right)^6\right]\end{aligned} \tag{4.35}$$

In the following figure we compare this result with the numerical renormalisation group method [21]. The definition of T_B is related to the usual Kondo temperature through a simple factor $T_B = \frac{2}{\pi}T_K$.

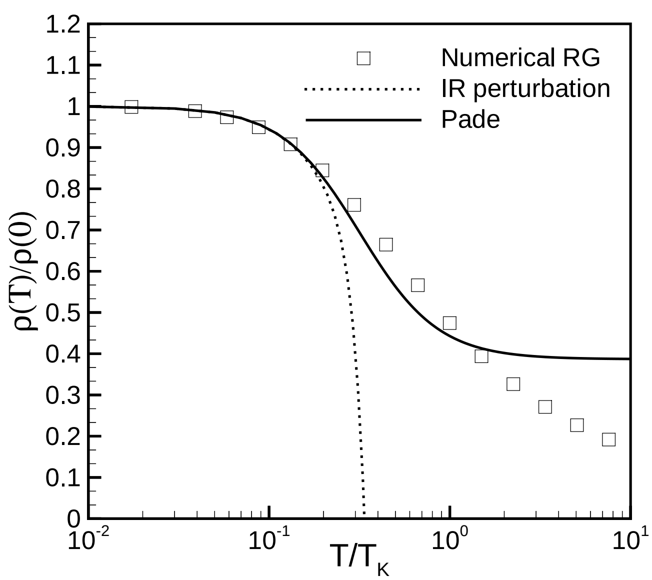

Fig. 2: Comparison with Numerical results.

The agreement is quite good considering that there is no fitting parameter. The Padé approximants were found to be very stable, and give a control of the curve $\rho(T)$ all the way to $T \approx T_K$, which is right in the crossover region. It is thus clear that our method provides a good analytical understanding of the strong coupling resistivity.

5. Another application: duality

The general structure of the IR hamiltonians is given by a set of local conserved quantities, plus at most one non local conserved quantity. This implies some duality properties that we now discuss.

5.1. Duality in Kondo with higher spin

The main thing about expression (3.24) is that it contains only *one* type of exponential [7]. Qualitatively, this is a consequence of integrability: it is natural to expect the trajectory to appear integrable both from the UV and IR fixed point; on the other hand, theories with several harmonics are generally non integrable - therefore, only one harmonic can occur. Quantitatively, this leads to a very strong similarity of the physical properties expanded near the UV or near the IR fixed point, after replacement $g \to \frac{1}{g}$; in particular, quantities that are "blind" to the integer spin conserved quantities, if any, will exhibit a complete *duality* symmetry between the UV and the IR.

To discuss the matter further, let us compute the boundary free energy at vanishing temperature and with an applied field $2HS_z$ (S_z taking values $j, j-1, \ldots, -j$ in the representation of spin j). We introduce the quantity ϵ_{2j} defined by

$$\tilde{\epsilon}_{2j} = \int \frac{d\omega}{2\pi} \frac{\sinh\left[\left(\frac{1}{1-g} - n\right)\frac{\pi\omega}{2}\right]}{\sinh\left[\frac{g}{1-g}\frac{\pi\omega}{2}\right]} \tilde{\epsilon}(\omega) - 2V(2j-1). \tag{5.1}$$

Here ϵ is the quantity defined in eq. (6.9) of [3] with the conventions of the appendix ($M = 2, \hbar = 1, V \equiv 2V, e = 1$); one has in particular, $\epsilon_1 = \epsilon$ of [3]. The parameter V

[7] Here, we should stress that this is a regularization dependent feature, that holds for our dimensionnally regularized approach. Ohter exponentials would appear as counter terms in other approaches.

is related with the field by $H = gV$. One can then establish, from the well known TBA formula in the limit $T \to 0$, that (this generalizes slightly [22]. See also [7],[8])

$$\begin{aligned} f =& \int \frac{d\beta}{2\pi} \frac{1}{\cosh(\beta - \beta_B)} \epsilon_{2j}(\beta) \\ =& V \int \frac{d\omega}{2\pi} e^{i\omega(A-\beta_B)} \frac{1}{2\cosh\frac{\pi\omega}{2}} \frac{\sinh\left(\frac{1}{1-g} - 2j\right)\frac{\pi\omega}{2}}{\sinh\frac{g}{1-g}\frac{\pi\omega}{2}} \frac{G_-(\omega)G_+(0)}{\omega(\omega - i)} - V(2j-1). \end{aligned} \tag{5.2}$$

In this formula,

$$G_-(\omega) = \sqrt{\frac{2\pi}{g}} \frac{\Gamma\left[i\omega/2(1-g)\right]}{\Gamma\left[i\omega g/2(1-g)\right]\Gamma\left[1/2 + i\omega/2\right]} e^{i\omega\Delta}, \tag{5.3}$$

and $\Delta = \frac{1}{2}\ln\frac{1-g}{g} + \frac{1}{2(1-g)}\ln g$. To compute f, we close the contour in the upper half plane when $A > \beta_B$. The only poles are those at $\omega = 2(1-g)ni$, n a positive integer. The UV expansion of f follows

$$\begin{aligned} f =& V\sqrt{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{2nj+n}}{nn!} \frac{\sin 2jn\pi g}{\sin 2n\pi g} \frac{e^{-2(1-g)n\Delta}}{\Gamma(-ng)\Gamma\left[3/2 - n(1-g)\right]} \left(\frac{T_B}{e^A}\right)^{2n(1-g)} \\ & - 2jVg. \end{aligned} \tag{5.4}$$

We now recall the correspondence between the cut-off A and the physical field in that case $e^A = V\frac{G_+(0)}{G_+(i)}$. Using that the field coupled to the impurity in the Kondo problem is $H = gV$, together with the correspondence between the bare coupling λ and T_B, one has

$$\frac{e^A}{T_B} = \frac{He^{\Delta}}{\left[\lambda\Gamma(1-g)\right]^{1/1-g}}. \tag{5.5}$$

This allow us to rewrite the free energy in the form

$$\begin{aligned} f =& \frac{\sqrt{\pi}H}{g} \sum_{n=1}^{\infty} \frac{(-1)^{2nj+n}}{nn!} \frac{\sin 2jn\pi g}{\sin 2n\pi g} \frac{1}{\Gamma(-ng)\Gamma\left[3/2 - n(1-g)\right]} \left[\frac{\lambda\Gamma(1-g)}{H^{1-g}}\right]^{2n} \\ & - 2jH. \end{aligned} \tag{5.6}$$

When $A < \beta_B$ on the other hand, we close the contour in the lower half plane. There are now two types of poles: the ones at $\omega = -(2n+1)i$ give contributions to the free energy of the form $\left(\frac{e^A}{T_B}\right)^{2n+1}$, while those of the form $\omega = -2ni\frac{1-g}{g}$ give the contribution (which we will refer to as "non-analytic")

$$\begin{aligned} f_{non-analytic} =& \sqrt{\pi}H \sum_{n=1}^{\infty} \frac{(-1)^{2nj}}{nn!} \frac{\sin(2j-1)n\pi/g}{\sin 2n\pi/g} \frac{1}{\Gamma(-n/g)\Gamma\left[3/2 - n(1-1/g)\right]} \\ & \left[\frac{\lambda\Gamma(1-g)}{H^{1-g}}\right]^{-\frac{2n}{g}} - (2j-1)\frac{H}{g}. \end{aligned} \tag{5.7}$$

From this we deduce the relation (to be used in (3.24))

$$\lambda_d = \frac{\sin\frac{\pi}{g}}{\pi g}\Gamma\left(\frac{1}{g}\right)\left[\frac{\sin\pi g}{\pi}g\Gamma(g)\right]^{\frac{1}{g}}\lambda^{-\frac{1}{g}}, \tag{5.8}$$

together[8] with

$$f\left(j,\lambda,H,g\right) \equiv f\left(j-\frac{1}{2},\lambda_d,\frac{H}{g},\frac{1}{g}\right), \tag{5.9}$$

where the equality holds up to analytical terms (odd powers) in H/T_B.

This duality has an obvious physical origin. We can compute the free energy near the UV fixed point perturbatively in powers of λ, or near the IR fixed point perturbatively in powers of $\lambda_d \propto \lambda^{-1/g}$ and in powers of $\lambda^{-1/(1-g)}$. The firs type of terms comes from the Kondo type interaction near the IR fixed point, that looks formally like the one near the UV fixed point, but with the replacements $j \to j-1/2$, $g \to 1/g$ and $H \to H/g$. It is interesting to discuss the later replacement in more details - the physical interaction near the UV and IR fixed points of course does not change, it is always $2HS_z$. However, to take this into account in the integrable approach, one needs to trade this term for a shift of the field ϕ in the Kondo interaction: the way this trading takes place depends on the charge of the exponentials, and this is why there is a rescaling in the TBA expressions, which are formally computed using an action with a H dependent Kondo coupling: see [23] and below for more details. Since all the integrals near the IR fixed point are defined by analytical continuation, they clearly lead to results obeying (5.9). In addition, the local and non local integrals of motion commute: therefore, in physical properties that involve the logarithm of the partition function (for instance), the terms coming from the Kondo type perturbation near the IR fixed point *do not* mix with the terms coming from the integer spin conserved quantities [9]. Therefore, to the non analytic contribution to f near the IR fixed point, is simply *added* an analytic contribution in odd powers of H/T_B. The structure of this analytic contribution is actually extremely simple, and depends only weakly on the spin.

Of course, the argument establishing the duality also holds at non vanishing temperature. Even though no close expression is known for the free energy in that case, we thus expect (5.9) to still hold, this time up to terms analytical in powers of $H/T_B, T/T_B$.

---

[8] Observe that (5.8) is very similar to (3.23). It would become identical if the the boundary sine-Gordon term came with the coupling $2\lambda\sin\pi g$, which is actually the natural choice within the quantum group framework underlying these problems.

[9] This remarkable property is clearly visible on the logarithms of the reflectiom matrix elements; see the formula (3.19) for a completely analogous example in the context of the boundary sine-Gordon model.

That (3.22) contains only *one* cosine has a simple physical meaning here - the flow approaches the IR fixed point along a direction where there is a term in the hamiltonian corresponding to tunneling of electrons, but no term for tunneling of pairs, triplets etc.

The most interesting properties to study in that context are transport properties, for which a non equilibrium formalsim such as Keldysh is required. Some of our conventions are discussed in the appendix; here, we will concentrate on the salient features only. Introducing a vector potential $a(t)$, the current is computed by $I(t) = \frac{\delta \ln Z}{\delta a}$, and expanded perturbatively near the UV or IR fixed point. Consider the UV fixed point first: there, the potential vector can be reabsorbed into the cosine term by a shift of the boson, so, restricting to constant voltage V, the current expands as a series of Coulomb gas integrals somewhat similar to the ones in equilibrium; the key difference however, is that the vertex operators $V_\pm = \exp \pm i\sqrt{2\pi g}\Phi$ are integrated on the Keldysh contour, represented in figure 3, and that contour ordered propagators are used.

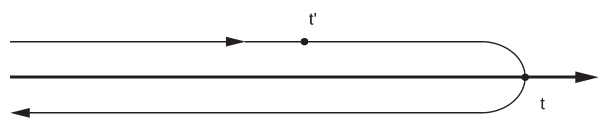

Fig. 3: Keldysh contour.

More specifically, a vertex operator stands at the extremity of the contour at time t, while other operators are integrated on the contour. The only non vanishing contributions are the ones which are electrically neutral. At non vanishing voltage V, each vertex operator $V_\pm(t')$ comes with an additional phase $e^{i\pm gVt'}$. The contour ordered propagator is such that $\langle T_c \left[\Phi(t')\Phi(t'')\right]\rangle = -\frac{1}{2\pi}\ln(t_> - t_<)$, where $t_>$ denotes the time that is the latest as measured along the contour, and $t_<$ the time that is the earliest. For instance, for t' on the above or lower part of the contour as in figure 3, the contraction that appears in the computation of the current is, $\langle T_c \left[e^{i\sqrt{2\pi g}\Phi(t)}e^{-i\sqrt{2\pi g}\Phi(t')}\right]\rangle = \frac{1}{(t-t')^{2g}}$, resp. $\frac{1}{(t'-t)^{2g}}$. The non trivial monodromy of the vertex operators ensures that the contribution of the two parts of the contour do not cancel out, and a non trivial result is obtained (see the appendix for some examples). At finite temperature T, the only change is that, in the propagator, $\ln t$ is replaced by $\ln \frac{\sinh \pi T t}{\pi T}$.

Now, our point is not so much to discuss the structure of this expansion (many details on this issue can be found in [24] for instance), but to comment on the duality properties it might give rise to. For this, let us investigate the computation of the current in

the IR. If, in our framework of dimensional regularization, the only operator in the IR were the $\cos\sqrt{\frac{2\pi}{g}}\tilde{\Phi}$, duality would easily follow from the matching of the two expansions. The complication we have to discuss is the role of all the $\mathcal{O}_{2k+2}$ operators added to the hamiltonian.

Let us first make a crucial observation. Consider for instance the contour ordered propagator in the two situations of figure 4,

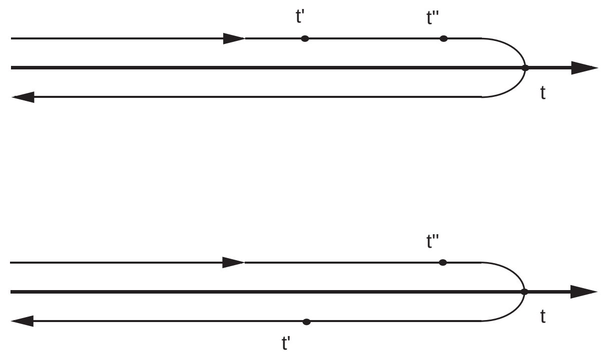

Fig. 4: Conserved quantity on the contour.

where say $\partial\Phi(t')$ (we call in this paragraph Φ what is in fact the dual of the original field $\tilde{\Phi}$ for notational simplicity; ∂ denotes time derivative) is inserted at time t', and some expression $X(t'')$ at time t''. In the first situation where t' occurs earlier on the contour,

$$\langle T_c\left[\partial\Phi(t')X(t'')\right]\rangle = \sum_{contractions} \frac{1}{2\pi(t''-t')}\hat{X}(t''),$$

where $\hat{X}$ denotes the remainder in X once contracted. In the second situation where t' occurs later on the contour,

$$\langle T_c\left[\partial\Phi(t')X(t'')\right]\rangle = \sum_{contractions} \frac{-1}{2\pi(t'-t'')}\hat{X}(t''),$$

and of course, the two expressions are actually equal. This easily generalizes to cases where $\partial\Phi$ is replaced by any polynomial in derivatives of Φ, in particular the $\mathcal{O}_{2k+2}$. As for X, it can be one of the $\mathcal{O}_{2k+2}$ itself, as well as a product of such an operator by a vertex operator, the result still holds: in other words, the result of the contraction does not depend on the order on the contour, and for the $\mathcal{O}_{2k+2}$ operators, integrals along the Keldysh contour just behave like ordinary contour integrals - a somewhat trivial fact, once one remembers that there is no cut in the complex plane for the contractions involved here.

This observation being made, we observe that the $\mathcal{O}_{2k+2}$ in the IR hamiltonian give rise to two complications. First, when one trades the coupling of the vector potential $\int a\partial_t\Phi$

for a shift of the field Φ, this time not only does one get a shift $\frac{1}{g}Vt$ in the argument of the cosine; one also gets in the new hamiltonian additional terms made of a and polynomials in the derivatives of Φ (for instance, the $:(\partial\Phi)^4:$ in $:T^2:$ gives rise to a $a^2:(\partial\Phi)^2:$ term, etc). Thus, when defining the current as the functional derivative of $\ln Z$ with respect to a, one gets, in the IR, a more complicated expression than in the UV: what has to be inserted at t on the Keldysh contour is the sum of a vertex operator and a series of polynomials in derivatives of Φ.

Now consider the perturbative computation of this current in the IR: we have to insert on the Keldysh contour either vertex operators or operators $\mathcal{O}_{2k+2}$. For the component of the current at t that is not the vertex operator however, no cut is necessary at t. According to the observation above, the integrals on the Keldysh contour of the various insertions then just behave like ordinary integrals, for which the upper and lower parts of the contour cancel out - in other words, the current is still obtained by only inserting vertex operators at t, in complete analogy with the UV case.

The second complication due to the $\mathcal{O}_{2k+2}$ is that these operators contribute to the perturbation series in the IR. Consider thus a generic term in the perturbation series, where a few vertex operators as well as conserved quantities have been inserted. To regulate divergences, it might be necessary to slightly displace the contours - this does not matter anyway, as we now argue. Indeed, consider moving the contours for the insertions of conserved quantities, say $\mathcal{O}_{2k+2}$ and $\mathcal{O}_{2k'+2}$. Since they are polynomials in derivatives of Φ, according to our observation above, these contours can be deformed as for usual integrals. The residue of their short distance expansion is a total derivative, so when we move one contour through the other, we are left with the contour integral of a total derivative. If in turn we try to deform this contour to zero, since the short distance expansion of a total derivative with any quantity cannot have a simple pole, no obstacle is met. In other words, we can freely pass through one another the contours for conserved quantities $\mathcal{O}_{2k+2}$.

Let us now try to pass these contours through the vertex operators. Consider thus a situation as the one in figure 5 where we have four vertex operators inserted on the Keldysh contour, and are trying to pass the $\mathcal{O}_{2k+2}$ contour through them.

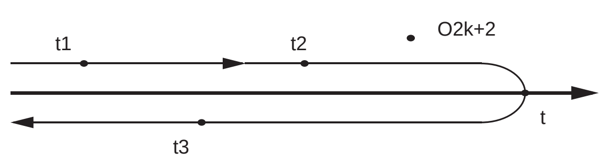

Fig. 5: Vertex operators.

In doing so, we encouter four poles, whose residues are total derivatives. Let us call the residue of the expansion of $\mathcal{O}_{2k+2}$ and V_ϵ, $\partial\mathcal{O}_{k,\epsilon}$. If t_1, t_2, t_3, t are the arguments of the four vertex operators, the total quantity picked up is

$$\partial_{t_1}\mathcal{O}_{k,\epsilon_1}V_{\epsilon_2}(t_2)V_{\epsilon_3}(t_3)V_\epsilon(t) + \text{ permutations}$$

Instead of contour integrating this quantity, let us simply look at its contour ordered average. Because the various contractions depend only on the *difference* of arguments, the effect of summing over permutations is to compute the derivatives with respect of sums of arguments of quantities that depend only on their differences, that is, is zero. Hence, the contribution of the residues when moving the $\mathcal{O}_{2k+2}$ contour through the vertex insertions cancels out, and we can squeeze this contour to zero. In other words, the $\mathcal{O}_{2k+2}$ *do not* contribute to the current at all. This is independent of the voltage or the temperature. As far as the current goes therefore, it is fully determined, in the scheme where integrals are analytically regularized, by the $2\lambda\cos\sqrt{2\pi g}\Phi$ perturbation in the UV and $2\lambda_d\cos\sqrt{\frac{2\pi}{g}}\tilde{\Phi}$ in the IR. This allows us to conclude that

$$I(\lambda, g, V, T) = gV - gI\left(\lambda_d, \frac{1}{g}, gV, T\right). \tag{5.10}$$

6. Conclusions

In conclusion, we would like to stress that the implementation of the IR perturbation theory, as well as the existence of duality, rely completely on the integrability of the problem. The latter acts as a symmetry that restricts the IR hamiltonian in a drastic fashion, so that the structure of the IR perturbation is almost the same as the UV one, maybe up to analytical terms. In general impurity problems, we do not expect the duality to be more than a quick qualitative argument to find the leading irrelevant operator. We also do not expect IR perturbation theory to make much sense, because of the difficulty in regularizing higher order terms when the operators do not commute.

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Appendix A. Normalization of conserved quantities in the sine-Gordon model

We discuss here the problem of determining the constant λ_{2k+1} in the definition (3.4). To do so, we consider the free action in the bulk, to which we add two perturbations: one of them is a field coupled to the $U(1)$ charge, and the other a term proportional to the conserved quantity $\mathcal{I}_{2k+1}$. Going to a hamiltonian description in the closed string channel, we obtain

$$\mathcal{H} = \mathcal{H}_{free} + V\sqrt{\frac{2g}{\pi}}\int_{-\infty}^{\infty} dy\partial_y\Phi + \mu\int_{-\infty}^{\infty}\mathcal{O}_{2k+2}. \tag{A.1}$$

The question we then consider is the ground state energy of this theory. It can be computed using the integrable structure [25],[26],[3]. We use here the notations of section VI of the latter reference; in addition, we set $M = 2$, $\hbar = 1$, $V \equiv 2V$ and $e = 1$. We denote the rapidity by β instead of θ. The constant λ of this reference corresponds, in the present paper, to $\lambda \equiv \frac{1}{g} - 1$; it of course has nothing to do with what we call λ in the present paper, that is the bare coupling.

The problem factorizes into R and L components. Consider say the R sector. The equations determining the ground state density of particles depend only on the momenta, which are not affected by the perturbations of the hamiltonian. The cut-off A however changes, in a way that depends on the perturbations in a crucial way. The equivalent of eqn (6.9) of [3] is now

$$V - e^{\beta} - \mu\frac{\lambda_{2k+1}}{2}e^{(2k+1)\beta} = \epsilon(\beta) - \int_{-\infty}^{A}\Phi(\beta-\beta')\epsilon(\beta')d\beta'. \tag{A.2}$$

It follows, if the Fourier transforms are defined as in [3], and if we denote $\epsilon_-(\omega) = \tilde{\omega}e^{-i\omega A}$, that

$$\begin{aligned}\epsilon_-(\omega) = &-\frac{1}{i}\frac{G_-(\omega)G_+(i)}{\omega - i}e^{A} + \frac{V}{i}\frac{G_-(\omega)G_+(0)}{\omega}\\ &-\frac{\mu\lambda_{2k+1}}{2i}\frac{G_-(\omega)G_+[(2k+1)i]}{\omega-(2k+1)i}e^{(2k+1)A},\end{aligned} \tag{A.3}$$

where the kernels are given in (5.3) above, and in eqn (6.6) of [3]. The cutoff A is such that $\epsilon(A) = 0$ ie $\lim_{\omega\to\infty}\omega\epsilon_-(i\omega) = 0$. We will restrict ourselves to the case where $V, \mu >> 1$, where the first term in (A.3) becomes negligible. It then follows that

$$e^{(2k+1)A} = \frac{2V}{\mu\lambda_{2k+1}}\frac{G_+(0)}{G_+[(2k+1)i]}. \tag{A.4}$$

We can then compute the energy per unit length

$$E = 2\int_{-\infty}^{A} d\beta\rho(\beta)\left[e^{\beta} + \mu\lambda_{2k+1}e^{(2k+1)\beta} - V\right] \approx 2\mu\lambda_{2k+1} - 2V\tilde{\rho}(0)\tilde{\rho}[-(2k+1)i], \quad \text{(A.5)}$$

where again we neglected the first term, of order one. Using another result form [3]

$$\tilde{\rho}(\omega) = \frac{1}{2i\pi}\frac{G_-(\omega)G_+(i)}{\omega - i}e^{(i\omega+1)A}, \quad \text{(A.6)}$$

we get, after some algebra,

$$E = -\frac{1}{\pi}\frac{2k+1}{2k+2}V^{\frac{2k+2}{2k+1}}\left(\frac{2}{\mu\lambda_{2k+1}}\right)^{\frac{1}{2k+1}}\frac{G_+(i)G_+(0)^{\frac{2k+2}{2k+1}}}{G_+[(2k+1)i]^{\frac{1}{2k+1}}}. \quad \text{(A.7)}$$

On the other hand, we can compute the energy directly from the hamiltonian. In the integral of $\mathcal{O}_{2k+2}$, only the leading term contributes since all the others involve second or higher derivatives of ϕ, which vanish at the saddle point. Therefore one has, using the normalization in (3.8),

$$E = -2V\sqrt{\frac{2g}{\pi}}\frac{2k+1}{2k+2}V^{\frac{2k+2}{2k+1}}\left(\frac{2}{\mu}\right)^{\frac{1}{2k+1}}\left(\frac{\sqrt{\frac{2g}{\pi}}}{2k+2}\right)^{\frac{1}{2k+1}}\frac{1}{(-2\pi)^{\frac{k}{2k+1}}}. \quad \text{(A.8)}$$

From this, it finally follows that

$$\lambda_{2k+1} = \left(\frac{\pi}{g}\right)^{k}(k+1)!\left(\frac{\Gamma\left[\frac{1}{2(1-g)}\right]}{\Gamma\left[\frac{g}{2(1-g)}\right]}\right)^{2k+1}\frac{\Gamma\left[\frac{(2k+1)g}{2(1-g)}\right]}{\Gamma\left[\frac{2k+1}{2(1-g)}\right]}. \quad \text{(A.9)}$$

Appendix B. Some remarks on Keldysh and analytic continuation

In [27], a formula for the current was proposed

$$I = gV - ig\pi T\frac{\partial}{\partial\ln\lambda}\ln\frac{Z(p,\lambda)}{Z(-p,\lambda)}. \quad \text{(B.1)}$$

Here, $Z(p)$ is an analytic continuation of the partition function at "imaginary voltage", accomplished through a Jack polynomials expansion [27].

The partition function at imaginary voltage is defined as $\mathrm{Tr}e^{-\mathcal{H}(p)/T}$, where p is an integer and $\mathcal{H}(p)$ is obtained from $\mathcal{H}$ by shifting the argument of the exponential $\cos\sqrt{2\pi g}\Phi \to \cos\left(\sqrt{2\pi g}\Phi + 2\pi pTy\right)$. The physical voltage is such that $2\pi pT = igV$, so

an analytical continuation in p from integer to imaginary values has to be carried out. The Keldysh formalism actually tells us how to perform this continuation. To see this, let us first recall some basic results. We consider the boundary sine-Gordon model with a vector potential $a(y)$ [2]. After the usual shift, one can write the partition function

$$Z = Z_0 \left\{ 1 + \lambda^2 \int_0^{1/T} \int_0^{1/T} dy_1 dy_2 \cos[a(y_1) - a(y_2)] p(|y_1 - y_2|) + \dots \right\}, \tag{B.2}$$

where dots stand for higher order terms, we restrict to $g < \frac{1}{2}$ so the integrals are all convergent, and

$$p(y) = \left(\frac{\pi T}{\sin \pi T y} \right)^{2g}. \tag{B.3}$$

The current then follows from $I = \frac{\delta \ln Z}{\delta a}$, together with the usual contour deformation

$$\begin{aligned} I(t) &= 2\lambda^2 \int_C \frac{dt'}{i} \sin[a(t) - a(t')] \langle T_c e^{i\sqrt{2\pi g}\Phi(t)} e^{-i\sqrt{2\pi g}\Phi(t')} \rangle + \dots \\ &= 2\lambda^2 \int_{-\infty}^{t} dt' \sin[a(t) - a(t')] \frac{P^{>}(t-t') - P^{<}(t-t')}{i} + \dots, \end{aligned} \tag{B.4}$$

where $P^{>}(resp. P^{<})$ is the analytic continuation to $y = it (resp. y = -it)$ of P. Let us now restrict to a DC voltage $a(t) = gVt$; in that case,

$$I = \lambda^2 \frac{P(gV) - P(-gV)}{i}, \tag{B.5}$$

where

$$P(x) = \int_0^{\infty} dt e^{ixt} \frac{P^{>}(t) - P^{<}(t)}{i}. \tag{B.6}$$

One has

$$P(x) = 2(\pi T)^{2g-1} \sin \pi g \int_0^{\infty} e^{ixt/\pi T} \frac{dt}{(\sinh t)^{2g}},$$

The latter integral is tabulated, so one gets

$$P(gV) = (2\pi)^{2g} T^{2g-1} \frac{\sin \pi g}{\sin\left(\pi g - \frac{igV}{2T}\right)} \frac{\Gamma(1-2g)}{\Gamma\left(1 - g + \frac{igV}{2\pi T}\right) \Gamma\left(1 - g - \frac{igV}{2\pi T}\right)}. \tag{B.7}$$

Let us now get back to the question of analytically continuing $Z(p)$ in (B.1). Consider the case p is an integer, so the quantities $Z(p)$ are well defined. Of course, for p integer, Z is *even* in p, so the argument of the derivative in (B.1) is identically zero. Let us nevertheless

expand it formally in powers of λ. At lowest order, the contribution to the current involves the quantities

$$Q(p) = \int_0^{1/T} \cos(2\pi pTy)p(y)dy. \tag{B.8}$$

One has

$$Q(p) = (\pi T)^{2g-1} \int_0^{\pi} e^{2ipy} \frac{dy}{(\sin y)^{2g}}$$

This integral is tabulated, and, for p an integer, reads

$$Q(p) = (2\pi)^{2g} T^{2g-1} (-1)^p \frac{\Gamma(1-2g)}{\Gamma(1-g+p)\Gamma(1-g-p)}. \tag{B.9}$$

Still for p an integer, the expressions (B.7) and (B.9) coincide: this is because, in that case, the voltage plays the role of a Matsubara frequency, so what we have here is the standard identity between Fourier transforms of temperature Green functions and retarded time-Green functions [28]. But we see now how to perform the continuation of $Z(p)$: we first need to deform the contour, at each order in peturbation theory, from the imaginary time interval to the Keldysh contour - this is possible for p an integer - and *then* replace p by $igVT/2\pi$ in the integrand.

At the level of final expressions, that is (B.7) and (B.9), however, it is less clear what must be done. The "recipe" proposed in [27] consists in expanding the integral Q into a sum of rational functions of Gamma functions

$$Q(p) = (2\pi)^{2g} T^{2g-1} \sum_{n=0}^{\infty} \frac{\Gamma(g+n)\Gamma(g+n+p)}{\Gamma^2(g)\Gamma(1+n)\Gamma(1+n+p)}, \tag{B.10}$$

and then perform the continuation simply by replacing p by the appropriate non integer value in each of the Gamma functions. The sum (B.10) was studied in details in [27], where it was shown that, for arbitrary p, this continuation of Q coincides with (B.7). Therefore, this gives the same result as the one obtained by deforming contour, which is the correct physical one, based on the Keldysh analysis.

Notice that the continuation using the expression (B.10) is not the same as the continuation discussed in [29]. In the latter work, the author expands the partition functions $Z(p)$ in (B.1) over Matsubara propagators, and then performs the continuation in p in each term independently. There is no reason why this definition should coincide with ours, and herefore it is not surprising that disagreements are found in [29]: only *one* continuation is

expected to work, and the conjecture made in [27] is that, for the tunneling problem described by the boundary sine-Gordon model, it is the one using the expansions (B.10), and for higher order, the corresponding sums based on Jack polynomials theory (see below).

To understand better why this might be true, we observe that $P(gV)$ has a simple power law behaviour $P \propto (V/T)^{2g-1}$ as $V/T \to \infty$, while $Q(p)$ in (B.9) does not. The power law behaviour is expected from the Keldysh contour representation, and on physical grounds as well: it is necessary for the current to have a finite expansion in terms of V/T_B as $T \to 0$. On the contrary (B.10), supplemented by the naive replacement of p by $\frac{igV}{2\pi T}$ does have the right behaviour.

When one considers higher powers of λ in the expansion of the current using the Keldysh formalism, one gets integrals which still coincide, for p integer, with the integrals occuring in $Z(p)$, through contour deformation. The challenge, if one whishes to prove the conjecture (B.1), is to show that replacing p by $igVT/2\pi$ in the deformed contour integrals *coincides* with replacing p by $igVT/2\pi$ in the expansion

$$Q_{2n}(p) = (2\pi)^{2g} T^{2ng-2n+1} \frac{1}{\Gamma(g)^{2n}} \sum_{\mathbf{m}} \prod_{i=1}^{n} \frac{\Gamma\left[m_i + g(n-i+1)\right]}{\Gamma\left[m_i + 1 + g(n-i)\right]} \times \\ \times \frac{\Gamma\left[p + m_i + g(n-i+1)\right]}{\Gamma\left[p + m_i + 1 + g(n-i)\right]}, \tag{B.11}$$

(where the sum is over all sets $\mathbf{m} = (m_1, \ldots, m_n)$ with $m_1 \geq m_2 \ldots \geq m_N \geq 0$), a result we explicitely checked above at lowest order. Clearly, the two procedures define analytical continuations of functions defined for integers. Under reasonable assumptions, it is known that two functions that coincide on integers and have the same behaviour at infinity are actually identical. Therefore, a proof of (B.1) would simply be that the continuation of (B.11) behaves, when $p \to \pm i\infty$, as $p^{2ng-2n+1}$. Though we have not proven this analytically, we have checked it numerically for the first few values of n.

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